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The path integration of a relativistic particle on a D-dimensional sphere

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Abstract. The fixed-energy amplitude of a relativistic particle near and on the sphere in D dimensions is given by the path integral approach. The Duru–Kleinert equivalence between the project amplitude of a relativistic particle near the surface of spheres in $D = 3$ and 4 dimensions with the Rosen–Morse and general Rosen–Morse systems are discussed.

1. Introduction

It is a simple problem to find out the wavefunctions and energy spectra of a particle moving on a sphere in D dimensions in the Schrödinger theory. Surprisingly, the path integral approach (PIA) of the same system was only completely solved in the past ten years [1–4]. The reason is that the surface of the sphere is a curved Riemannian space with a constant intrinsic curvature. The right quantization results for the angular variables on a sphere correspond to the group quantization rules (for reviews, see [1]). Therefore the correct rules for path integration in curved spaces should agree with these rules. For this, some PIA methods in curve space have been proposed in the past [1, 2, 5–12]. A consistent quantum equivalent principle (QEP) [1, 13–16] for the PIA in curved space was proposed in 1989 [1]. It can be used to construct a consistent new PIA in agreement with correct group quantization rules in space with curvature and torsion. Moreover, one can obtain the DK-transformation [1] based on this theory. The Green function with correct energy spectra was obtained using the new PIA in [17] with QEP. In this paper we apply the QEP to the PIA of a relativistic particle on a sphere in D dimensions. The correct energy spectra are recovered by QEP. We shall proceed in two steps [1]. First we shall use the experience gained using the standard Feynman path integral with angular decompositions of the time-sliced path integral in Euclidean space to introduce and solve an auxiliary time-sliced path integral involving only angular variables. This turns out to be very closely related to the desired path integral on the surface of the sphere. In the second step we shall correct the relativistic path integral by QEP so it ultimately describes the motion on the sphere. As an application of the relativistic PIA near a sphere we will discuss the DK-equivalence between the Rosen–Morse and general Rosen–Morse systems with the amplitudes of a relativistic particle near spheres in $D = 3$, 4 dimensions. The similar equivalence for the non-relativistic PIA near a sphere was given in [18, 19].

This paper is organized as follows. In section 2, we briefly review the formulation of the path integral for the relativistic potential problems in general affine space. We calculate

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the path integral of a relativistic spinless particle near and on the surface of a sphere in D dimensions in section 3. In section 4 we discuss the DK-equivalence between the Rosen–Morse and general Rosen–Morse systems with the relativistic project amplitude of spheres in $D = 3$, 4 dimensions. Our conclusions are summarized in section 5.

2. Path integral for relativitistic particle orbits

Let us first consider a point particle of mass M moving through a $(D+1)$-dimensional Minkowski space at a relativistic velocity. By using $t = -\mathrm{i}\tau = -\mathrm{i}x^4/c$, its path integral representation of the fixed-energy amplitude is conveniently formulated in a $(D+1)$-Euclidean spacetime with the Euclidean metric

$$(g_{\mu\nu}) = \mathrm{diag}(1, \ldots, 1, c^2) \tag{1}$$

and it is given by [1, 22]

$$(\boldsymbol{x}_b|\boldsymbol{x}_a)_E = -\frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}L \int D\rho\,\Phi[\rho]\int D^D x\,\mathrm{e}^{-A_E/\hbar} \tag{2}$$

with the action integral

$$A_E = \int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda\left[\frac{M}{2\rho\,(\lambda)}\boldsymbol{x}'^2\,(\lambda) - \rho\,(\lambda)\,\frac{(E-V)^2}{2Mc^2} + \rho\,(\lambda)\,\frac{Mc^2}{2}\right] \tag{3}$$

where L is the total invariant length of the path

$$L = c\int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda\,\rho(\lambda) \tag{4}$$

$\rho(\lambda)$ is an extra dimensionless fluctuating variable and $\Phi[\rho]$ is an appropriate gauge-fixing functional, for instance, choosing $\Phi[\rho] = \delta[\rho - 1]$ to fix $\rho(\lambda)$ being unity everywhere [1]. $\hbar/Mc$ is the well known Compton wavelength of a particle of mass M, $(E - V)$ is the system kinetic energy, and $\boldsymbol{x}$ is the spatial part of a $(D+1)$-vector $x = (\boldsymbol{x}, \tau)$ with invariant length $x = \sqrt{\boldsymbol{x}^2 + c^2\tau^2}$. This path integral forms the basis for studying relativistic potential problems. To obtain a tractable path integral for the potential V, we perform a f-transformation (e.g. [1, 20, 21])

$$\mathrm{d}\lambda = \mathrm{d}s\; f_l(\boldsymbol{x}_n) f_r(\boldsymbol{x}_{n-1}) \tag{5}$$

where we have supposed that $f_l(\boldsymbol{x})$ and $f_r(\boldsymbol{x})$ are invertible but otherwise arbitrary functions. The freedom in choosing $f_{l,r}$ amounts to an invariance under path-dependent λ-reparametrizations of the fixed-energy amplitude in (2). By this transformation, the $(D+1)$-dimensional relativistic fixed-energy amplitude for arbitrary time-independent potential turns into

$$(\boldsymbol{x}_b|\boldsymbol{x}_a)_E \approx -\frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}S\,\frac{f_l(\boldsymbol{x}_a) f_r(\boldsymbol{x}_b)}{\sqrt{2\pi\hbar\epsilon_b^s\rho_b f_l(\boldsymbol{x}_b) f_r(\boldsymbol{x}_a)/M}^D}\prod_{n=1}^{N+1}\left[\int \mathrm{d}\rho_n\,\Phi(\rho_n)\right]$$
$$\times\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}^D x_n}{\sqrt{2\pi\hbar\epsilon_n^s\rho_n f(\boldsymbol{x}_n)/M}^D}\right]\mathrm{e}^{-\hbar^{-1}A^N} \tag{6}$$

with the s-sliced action

$$A^N = \sum_{n=1}^{N+1}\left[\frac{M(\boldsymbol{x}_n - \boldsymbol{x}_{n-1})^2}{2\epsilon_n^s\rho_n f_l(\boldsymbol{x}_n) f_r(\boldsymbol{x}_{n-1})} - \epsilon_n^s\rho_n f_l(\boldsymbol{x}_n) f_r(\boldsymbol{x}_{n-1})\frac{(E-V)^2}{2M}\right.$$
$$\left. + \,\epsilon_n^s\rho_n f_l(\boldsymbol{x}_n) f_r(\boldsymbol{x}_{n-1})\frac{Mc^2}{2}\right]. \tag{7}$$

In the measure, we have used the abbreviation $f(\boldsymbol{x}_n) = f_l(\boldsymbol{x}_n) f_r(\boldsymbol{x}_n)$. The sign $\approx$ becomes an equality for $N \to \infty$. By shifting the product index and the subscripts of f_n by one unit, and by compensating for this with a prefactor, the integration measure in (6) acquires the postpoint form

$$\int_0^\infty \mathrm{d}S \frac{f_l(\boldsymbol{x}_a) f_r(\boldsymbol{x}_b)}{\sqrt{2\pi\hbar\epsilon_b^s \rho_b f_l(\boldsymbol{x}_b) f_r(\boldsymbol{x}_a)/M}^D} \sqrt{\frac{f(\boldsymbol{x}_b)}{f(\boldsymbol{x}_a)}}^D \prod_{n=1}^{N+1}\left[\int \mathrm{d}\rho_n\, \Phi(\rho_n)\right]$$
$$\times \prod_{n=2}^{N+1}\left[\int_{-\infty}^{\infty} \frac{\mathrm{d}^D \Delta x_n}{\sqrt{2\pi\hbar\epsilon_n^s \rho_n f(\boldsymbol{x}_n)/M}^D}\right]. \quad (8)$$

The integrals over each coordinate difference $\Delta x_n = x_n - x_{n-1}$ are performed at fixed postpoint positions x_n. We note that since x_n are Cartesian coordinates, the measures of integration in the s-sliced expressions (6) and (8) are certainly identical

$$\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} \mathrm{d}^D x_n\right] = \prod_{n=2}^{N+1}\left[\int_{-\infty}^{\infty} \mathrm{d}^D \Delta x_n\right]. \quad (9)$$

However, their images under a non-holonomic mapping [1, 23] are different so that the initial form of the s-sliced path integral is a matter of choice. In the space with curvature and torsion it has been proved by [1]. Only the right-hand side of (9) gives the properly correct results. To simplify the subsequent discussion, it is preferable to work only with the postpoint regularization in which $f_l(\boldsymbol{x}) = f(\boldsymbol{x})$ and $f_r(\boldsymbol{x}) = 1$. Then the measure becomes simply

$$\int_0^\infty \mathrm{d}S \frac{f(\boldsymbol{x}_a)}{\sqrt{2\pi\hbar\epsilon_b^s \rho_b f(\boldsymbol{x}_a)/M}^D} \prod_{n=1}^{N+1}\left[\int \mathrm{d}\rho_n\, \Phi(\rho_n)\right] \prod_{n=2}^{N+1}\left[\int_{-\infty}^{\infty} \frac{\mathrm{d}^D \Delta x_n}{\sqrt{2\pi\hbar\epsilon_n^s \rho_n f(\boldsymbol{x}_n)/M}^D}\right]. \quad (10)$$

To obtain a general formulation in affine space, we now introduce the coordinate transformation. In D dimensions, it is given by

$$x^i = h^i(q). \quad (11)$$

The differential mapping may be written as

$$\mathrm{d}x^i = \partial_\mu h^i(q)\, \mathrm{d}q^\mu = e^i{}_\mu(q)\, \mathrm{d}q^\mu. \quad (12)$$

Since we must find all terms that will eventually contribute to order ϵ, we expand $(\Delta\boldsymbol{x}_n)^2 = (\boldsymbol{x}_n - \boldsymbol{x}_{n-1})^2$ up to fourth order in $\Delta q_n^\mu = q_n^\mu - q_{n-1}^\mu$ [24, 25]. For the finite coordinate differences, the non-holonomic mapping of $\Delta\boldsymbol{x}_n$ is given by [1]

$$\Delta x^i = e^i{}_\alpha \dot{q}^\alpha \Delta\lambda = e^i{}_\alpha \left[\Delta q^\alpha - \frac{1}{2!}\Gamma_{\mu\nu}{}^\alpha \Delta q^\mu \Delta q^\nu + \frac{1}{3!}(\partial_\sigma \Gamma_{\mu\nu}{}^\alpha + \Gamma_{\mu\nu}{}^\tau \Gamma_{\{\sigma\tau\}}{}^\alpha)\right.$$
$$\left. \times \Delta q^\mu \Delta q^\nu \Delta q^\sigma + \cdots\right]. \quad (13)$$

For brevity we have omitted the argument q_n in the $e^i{}_\alpha$s as well as the subscripts n of Δq^μ. Here $e^i{}_\alpha$ and $\Gamma_{\mu\nu}{}^\alpha$ are evaluated at the postpoint, $\dot{q}^\alpha$ stands for $\mathrm{d}q^\alpha/\mathrm{d}\lambda$ and the curly bracket around the indices denotes their symmetrization. From this we obtain the transformed path integral [1]

$$(\boldsymbol{x}_b|\boldsymbol{x}_a)_E \approx -\frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}S \frac{f(q_a)}{\sqrt{2\pi\hbar\epsilon_b^s \rho_b f(q_a)/M}^D} \prod_{n=1}^{N+1}\left[\int \mathrm{d}\rho_n\, \Phi(\rho_n)\right]$$
$$\times \prod_{n=2}^{N+1}\left[\int_{-\infty}^{\infty} \frac{\mathrm{d}^D \Delta q_n g^{1/2}(q_n)}{\sqrt{2\pi\hbar\epsilon_n^s \rho_n f(q_n)/M}^D}\right] \mathrm{e}^{-\hbar^{-1} A_{\mathrm{tot}}^N} \quad (14)$$

where the total s-sliced action

$$A_{\rm tot}^{N} = \sum_{n=1}^{N+1} A_{\rm tot}^{\epsilon}. \tag{15}$$

Each slice contains three terms

$$A_{\rm tot}^{\epsilon} = A^{\epsilon} + A_J^{\epsilon} + A_{\rm pot}^{\epsilon}. \tag{16}$$

The short-time postpoint action is given by [1]

$$\begin{aligned} A^{\epsilon}(q, q-\Delta q) = \frac{M}{2\epsilon^s \rho f}(\Delta x^i)^2 = \epsilon^s \rho f \frac{M}{2} g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} = \frac{M}{2\epsilon^s\rho f}\{g_{\mu\nu}\Delta q^{\mu}\Delta q^{\nu} \\ -\Gamma_{\mu\nu\lambda}\Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda} + [\tfrac{1}{3}g_{\mu\tau}(\partial_{\kappa}\Gamma_{\lambda\nu}{}^{\tau} + \Gamma_{\lambda\nu}{}^{\delta}\Gamma_{\{\kappa\delta\}}{}^{\tau}) + \tfrac{1}{4}\Gamma_{\lambda\kappa}{}^{\sigma}\Gamma_{\mu\nu\sigma}] \\ \times \Delta q^{\mu}\Delta q^{\nu}\Delta q^{\lambda}\Delta q^{\kappa} + \cdots\}. \end{aligned} \tag{17}$$

with the induced metric $g_{\mu\nu} = e^i{}_{\mu} e^i{}_{\nu}$ and the affine connection $\Gamma_{\lambda\kappa}{}^{\mu} = e_i{}^{\mu} e^i{}_{\kappa,\lambda}$. The Jacobian action A_J^{ϵ} which comes from the right-hand side of (9) can be expressed as [1]

$$-\frac{1}{\hbar}A_J^{\epsilon} = -\Gamma_{\{\mu\nu\}}{}^{\mu}\Delta q^{\nu} + \frac{1}{2}[\partial_{\{\mu}\Gamma_{\nu\kappa\}}{}^{\kappa} + \Gamma_{\{\nu\kappa}{}^{\sigma}\Gamma_{\{\sigma|\mu\}\}}{}^{\kappa} - \Gamma_{\{\nu\kappa\}}{}^{\sigma}\Gamma_{\{\sigma\mu\}}{}^{\kappa}]\Delta q^{\mu}\Delta q^{\nu} + \cdots \tag{18}$$

where the curly double brackets around the indices ν, μ, σ, κ indicate a symmetrization in τ and σ followed by a symmetrization in μ, ν, σ. The potential action $A_{\rm pot}^{\epsilon}$ has the form

$$-\epsilon^s \rho f \left(\frac{E - V(q)}{2M}\right)^2. \tag{19}$$

In these three equations, we have omitted the subindex n for convenience.

It is useful to re-express our result in a different form which clarifies the relation with the naive measure [1] of path integration

$$\prod_{n=1}^{N}\left[\int {\rm d}^D x_n\right] = \prod_{n=1}^{N}\left[\int {\rm d}^D q_n \sqrt{g(q_n)}\right]. \tag{20}$$

The correct measure of (14) can be expressed in terms of (20) as

$$\prod_{n=2}^{N+1}\left[\int {\rm d}^D \Delta q_n \sqrt{g(q_n)}\right] = \prod_{n=1}^{N}\left[\int {\rm d}^D q_n \sqrt{g(q_n)}\, {\rm e}^{A_{J_0}^{\epsilon}/\hbar}\right] \tag{21}$$

where $A_{J_0}^{\epsilon}$ is the Jacobian of the coordinate transformation from ${\rm d}^D x^i$ and ${\rm d}^D q^{\mu}$. It has the representation

$$-\frac{1}{\hbar}A_{J_0}^{\epsilon} = -\Gamma_{\nu\mu}{}^{\mu}\Delta q^{\mu} + \frac{1}{2}\partial_{\nu}\Gamma_{\nu\kappa}{}^{\kappa}\Delta q^{\mu}\Delta q^{\nu} + \cdots. \tag{22}$$

The corresponding expression for the s-sliced relativistic path integral (14) with the naive measure in the metric-affine space reads

$$\begin{aligned} (\boldsymbol{x}_b|\boldsymbol{x}_a)_E \approx -\frac{{\rm i}\hbar}{2Mc}\int_0^{\infty}{\rm d}S\, \frac{f(q_a)}{\sqrt{2\pi\hbar\epsilon_b^s\rho_b f(q_b)/M}^D}\prod_{n=1}^{N+1}\left[\int {\rm d}\rho_n\, \Phi(\rho_n)\right] \\ \times \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{{\rm d}^D q_n g^{1/2}(q_n)}{\sqrt{2\pi\hbar\epsilon_n^s\rho_n f(q_n)/M}^D}\right]{\rm e}^{-\hbar^{-1}A_{\rm tot}^N} \end{aligned} \tag{23}$$

with each slice in the total action $A_{\rm tot}^N$ contains three terms

$$A_{\rm tot}^{\epsilon} = A^{\epsilon} + \Delta A_J^{\epsilon} + A_{\rm pot}^{\epsilon} \tag{24}$$

where ΔA_J^ϵ is the difference between the correct and the wrong Jacobian actions in (18) and (22)

$$\Delta A_J^\epsilon = A_J^\epsilon - A_{J_0}^\epsilon. \tag{25}$$

Either (14) or (23) may be used as the correct path integral formulae in spaces with curvature and torsion.

In the absence of torsion, the affine connection reduces to the Riemann connection $\Gamma_{\{\mu\nu\}}{}^\lambda = \bar{\Gamma}_{\mu\nu}{}^\lambda$. It is easy to check that (25) simplifies to

$$-\frac{1}{\hbar}\Delta A_J^\epsilon = \frac{1}{6}\bar{R}_{\mu\nu}\Delta q^\mu \Delta q^\nu \tag{26}$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor. Being quadratic in Δq, the effect of the additional action can easily be evaluated perturbatively according to which $\Delta q^\mu \Delta q^\nu$ may be replaced by its lowest-order expectation

$$\langle \Delta q^\mu \Delta q^\nu \rangle_0 = \frac{\epsilon^s \hbar g^{\mu\nu}(q)}{M}. \tag{27}$$

Then ΔA_J^ϵ yields the additional effective potential

$$V_{\text{eff}} = -\frac{\hbar^2}{6M}\bar{R} \tag{28}$$

where $\bar{R}$ is the Riemann scalar curvature. By including this potential in the action, the relativistic path integral in a curved space can be written down in naive measure form as follows:

$$(\boldsymbol{x}_b|\boldsymbol{x}_a)_E \approx -\frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}S \frac{f(q_a)}{\sqrt{2\pi\hbar\epsilon_b^s\rho_b f(q_b)/M}^D}\prod_{n=1}^{N+1}\left[\int \mathrm{d}\rho_n\,\Phi(\rho_n)\right]$$
$$\times\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}^D q_n g^{1/2}(q_n)}{\sqrt{2\pi\hbar\epsilon_n^s\rho_n f(q_n)/M}^D}\right]\exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N+1}[A^\epsilon + A_{\text{pot}}^\epsilon + \epsilon^s V_{\text{eff}}]\right\}. \tag{29}$$

The integrals over q_n are performed successively downwards over $\Delta q_{n+1} = q_{n+1} - q_n$ at fixed q_{n+1}.

There is another equivalent s-sliced path integral representation which leads to the same D-dimensional relativistic physics in general metric affine spaces. Its postpoint form of the s-sliced path integral is given by [1]

$$(\boldsymbol{x}_b|\boldsymbol{x}_a)_E \approx -\frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}S \frac{f(q_a)}{\sqrt{2\pi\hbar\epsilon_b^s\rho_b f(q_a)/M}^D}\prod_{n=1}^{N+1}\left[\int \mathrm{d}\rho_n\,\Phi(\rho_n)\right]$$
$$\times\prod_{n=2}^{N+1}\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}^D \Delta q_n g^{1/2}(q_n)}{\sqrt{2\pi\hbar\epsilon_n^s\rho_n f(q_n)/M}^D}\right]\mathrm{e}^{-\hbar^{-1}\sum_{n=1}^{N+1}[A^\epsilon + A_J^\epsilon + A_{\text{pot}}^\epsilon]} \tag{30}$$

where each s-sliced action has the form

$$A^\epsilon + A_J^\epsilon + A_{\text{pot}}^\epsilon = \frac{M}{2\epsilon^s\rho f}g_{\mu\nu}\Delta q^\mu \Delta q^\nu - \frac{\hbar}{2}\Gamma_\mu{}^\mu{}_\nu \Delta q^\nu + \epsilon^s \rho f \frac{h^2}{8M}(\Gamma_\mu{}^\mu{}_\nu)^2$$
$$-\epsilon^s\rho f\left(\frac{E - V(q)}{2M}\right)^2. \tag{31}$$

Remarkably, this expression only involves the connection contracted in the first two indices

$$\Gamma_\mu{}^{\mu\nu} = g^{\mu\lambda}\Gamma_{\mu\lambda}{}^\nu. \tag{32}$$

The relativistic stable path integrals of (6) have a more elegant representation if the systems are in two-dimensional Minkowski space or rotationally invariant systems in any dimensions. For simplicity, we consider only rotationally invariant systems. The two-dimensional systems have the same effective potential. By decomposing the (6) into angular parts [1, 24, 26–28] and integrating over the angular part, we get

$$(\boldsymbol{x}_b|\boldsymbol{x}_a)_E = \frac{1}{(r_b r_a)^{(D-1)/2}} \sum_{l=0}^{\infty} (r_b|r_a)^f_{E,l} \sum_{\hat{m}} Y_{l\hat{m}}(\hat{\boldsymbol{x}}_b) Y^*_{l\hat{m}}(\hat{\boldsymbol{x}}_a) \tag{33}$$

where the functions $Y_{l\hat{m}}(\hat{\boldsymbol{x}})$ are the D-dimensional hyperspherical harmonics [29, 30] and $(r_b|r_a)^f_{E,l}$ is the purely radical amplitude. It has the s-sliced version

$$(r_b|r_a)^f_{E,l} \approx -\frac{\mathrm{i}\hbar}{2Mc} \int_0^\infty \mathrm{d}S \prod_{n=1}^{N+1} \left[\int \mathrm{d}\rho_n \, \Phi(\rho_n) \right] \frac{f_l(r_a) f_r(r_b)}{\sqrt{2\pi\hbar\epsilon^s_b \rho_b f_l(r_b) f_r(r_a)/M}} \times \prod_{n=1}^{N} \left[\int_0^\infty \frac{\mathrm{d}r_n}{\sqrt{2\pi\hbar\epsilon^s_n \rho_n f_n/M}} \right] \exp\left\{ -\frac{1}{\hbar} A^{N,f}_l[r, r'] \right\} \tag{34}$$

with the s-sliced action

$$A^{N,f}_l[r, r'] = \sum_{n=1}^{N+1} \left\{ \frac{M}{2\epsilon^s_n f_l(r_n) f_r(r_{n-1}) \rho_n} (r_n - r_{n-1})^2 - \rho_n \epsilon^s_n f_l(r_n) f_r(r_{n-1}) \hbar \log \tilde{I}_{D/2-1+l} \left(\frac{M r_n r_{n-1}}{\hbar \epsilon^s_n f_l(r_n) f_r(r_{n-1}) \rho_n} \right) - \epsilon^s_n f_l(r_n) f_r(r_{n-1}) \rho_n \frac{[E - V(r_n)]^2}{2Mc^2} + \epsilon^s_n f_l(r_n) f_r(r_{n-1}) \rho_n \frac{Mc^2}{2} \right\}. \tag{35}$$

The continuous representation has the form

$$(r_b|r_a)^f_{E,l} = -\frac{\mathrm{i}\hbar}{2Mc} \int_0^\infty \mathrm{d}S \int D\rho \, \Phi[\rho(s)] \int Dr(s) \exp\left\{ -\frac{1}{\hbar} A^f_l[r, \dot{r}] \right\} \tag{36}$$

where

$$A^f_l[r, \dot{r}] = \int_0^S \mathrm{d}s \left[\frac{M}{2\rho(s) f_l(r) f_r(r)} \dot{r}^2(s) + \rho(s) f_l(r) \times \left(\frac{\hbar^2}{2M} \frac{(l + D/2 - 1)^2 - 1/4}{r^2} - \frac{[E - V(r)]^2}{2Mc^2} + \frac{Mc^2}{2} \right) f_r(r) \right]. \tag{37}$$

From the action, we observe that the singular potential can be removed by the regulating function $f(r) = f_l(r) f_r(r)$. The action looks like the time-sliced version of the naively expected radial path integral in D dimensions. It is worth noting that one has to replace the original centrifugal term by a well-behaved one to make the result convergent when performing the path integration. The required replacement

$$\frac{\hbar^2}{2M} \frac{(l + D/2 - 1)^2 - 1/4}{r_n r_{n-1}} \to -\hbar \log \tilde{I}_{D/2-1+l} \left(\frac{M r_n r_{n-1}}{\hbar \epsilon_n \rho_n} \right). \tag{38}$$

was first introduced in [26].

We now perform a transformation to get a conventional kinetic term

$$A^N_0 = \sum_{n=1}^{N+1} \frac{M}{2\epsilon^s_n \rho_n} (\Delta q_n)^2. \tag{39}$$

This can be achieved by using the transformation function $r = h(q)$ with

$$h'^2(q) = f(r). \tag{40}$$

By applying this transformation to (37) and including the measure correction from the QEP, we get the DK-transform [1] of the s-sliced amplitude

$$(r_b|r_a)^{\rm DK}_{E,l} \approx -\frac{{\rm i}\hbar}{2Mc}\int_0^\infty {\rm d}S \prod_{n=1}^{N+1}\left[\int {\rm d}\rho_n\,\Phi(\rho_n)\right]\frac{f_b^{1/4} f_a^{1/4}}{\sqrt{2\pi\hbar\epsilon_b^s\rho_b/M}}$$
$$\times \prod_{n=1}^{N}\left[\int_0^\infty \frac{{\rm d}\Delta q_n}{\sqrt{2\pi\hbar\epsilon_n^s\rho_n/M}}\right]\exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon_n^s\rho_n}(\Delta q_n)^2 + \epsilon_n^s\rho_n f(q_n)\right.\right.$$
$$\left.\left.\times\left[\frac{\hbar^2}{2M}\frac{(l+D/2-1)^2-1/4}{r_n r_{n-1}} - \frac{[E-V(r_n)]^2}{2Mc^2} + \frac{Mc^2}{2}\right] + \epsilon_n^s\rho_n V_{\rm eff}\right\} \tag{41}$$

which has the continuum limit

$$(r_b|r_a)^{\rm DK}_{E,l} = \frac{-{\rm i}\hbar}{2Mc} f_b^{1/4} f_a^{1/4}\int_0^\infty {\rm d}S\int D\rho\,\Phi[\rho]\int Dq\;{\rm e}^{-A_s^{\rm DK}/\hbar} \tag{42}$$

with

$$A_s^{\rm DK} = \int_0^S {\rm d}s\left[\frac{M}{2\rho(s)}\dot q^2(s) + \rho(s) f(q(s))\right.$$
$$\left.\times\left(\frac{\hbar^2}{2M}\frac{(l+D/2-1)^2-1/4}{r^2(q(s))} - \frac{[E-V(r)]^2}{2Mc^2} + \frac{Mc^2}{2}\right) + \rho(s)V_{\rm eff}\right]. \tag{43}$$

The superscript DK in (43) indicates that the system has been performed by the DK-transformation. The effective potential $V_{\rm eff}$ is given by (e.g. [1, 31])

$$V_{\rm eff} = -\frac{\hbar^2}{M}\left[\frac{1}{4}\frac{h'''}{h'} - \frac{3}{8}\left(\frac{h''}{h'}\right)^2\right]. \tag{44}$$

By defining the q-space amplitude

$$(q_b|q_a) \equiv -{\rm i}\int_0^\infty {\rm d}S\int D\rho\,\Phi[\rho]\int Dq\;{\rm e}^{-A_s^{\rm DK}/\hbar} \tag{45}$$

we can represent (40) as

$$(r_b|r_a)^{\rm DK}_{E,l} \equiv \frac{\hbar}{2Mc} f_b^{1/4} f_a^{1/4}(q_b|q_a). \tag{46}$$

This equation is the DK-transform of an arbitrary relativistic time-independent potential problem. In the paper, we will use this transformation to evaluate the path integral of the Rosen–Morse system.

3. A relativistic mass point near and on a sphere

The time-sliced path integral of a relativistic mass point near a sphere can be given from (2) by restricting the radial variable r to the surface of a fixed radius R and identifying the unit vector $\hat{\boldsymbol{x}}$ with $\hat{\boldsymbol{u}}$. It has the λ-slicing form

$$(\boldsymbol{u}_b|\boldsymbol{u}_a)_E \approx \frac{-{\rm i}\hbar}{2Mc}\int_0^\infty {\rm d}L\prod_{n=1}^{N+1}\left[\int {\rm d}\rho_n\,\Phi(\rho_n)\right]\frac{1}{(2\pi\hbar\epsilon_b\rho_b/MR^2)^{(D-1)/2}}$$
$$\times\prod_{n=1}^{N}\left[\int\frac{{\rm d}\boldsymbol{u}_n}{(2\pi\hbar\epsilon_n\rho_n/MR^2)^{(D-1)/2}}\right]{\rm e}^{-A_E^N/\hbar} \tag{47}$$

with the sliced action

$$A_E^N = \sum_{n=1}^{N+1} \left\{ \frac{MR^2}{2\rho_n\epsilon_n}(\boldsymbol{u}_n - \boldsymbol{u}_{n-1})^2 - \rho_n\epsilon_n \left(\frac{E^2}{2Mc^2} - \frac{Mc^2}{2} \right) \right\}. \tag{48}$$

By decomposing the first term of (48) into angular parts [1–3], we have

$$\exp\left[-\frac{MR^2}{2\hbar\epsilon_n\rho_n}(\boldsymbol{u}_n - \boldsymbol{u}_{n-1})^2 \right] = \sum_{l=0}^{\infty} \tilde{a}_l(\varsigma_n) \sum_{\boldsymbol{m}} Y_{l\boldsymbol{m}}(\boldsymbol{u}_n) Y_{l\boldsymbol{m}}^*(\boldsymbol{u}_{n-1}) \tag{49}$$

where

$$\tilde{a}_l(\varsigma_n) = \left(\frac{2\pi}{\varsigma_n} \right)^{(D-1)/2} \tilde{I}_{D/2+l-1}(\varsigma_n) \qquad \varsigma_n = \frac{MR^2}{\hbar\epsilon_n\rho_n} \tag{50}$$

with $\tilde{I}_\nu(z)$ being defined by $\tilde{I}_\nu(z) = \sqrt{2\pi z}\, \mathrm{e}^{-z} I_\nu(z)$. The $Y_{l\boldsymbol{m}}(\boldsymbol{u}_n)$ denote the hyperspherical harmonics. The time-sliced path integral (47) of a relativistic particle near a sphere in D dimensions can be solved by applying the orthonormal relation of the hyperspherical harmonic and using the asymptotic behaviour [1, 30]

$$\tilde{I}_\nu(z) \xrightarrow{z\to\infty} 1 - \frac{m^2 - 1/4}{2z} + \cdots = \mathrm{e}^{(-m^2-1/4)/2z} + \cdots \tag{51}$$

to (47). The continuum limit is given by

$$(\boldsymbol{u}_b|\boldsymbol{u}_a)_E = \frac{\hbar}{2Mc} \sum_{l=0}^{\infty} \frac{2Mc^2\hbar\mathrm{i}}{E^2 - \{M^2c^4 + (c\hbar/R)^2[(D/2+l-1)^2 - 1/4]\}} \times \sum_{\boldsymbol{m}} Y_{l\boldsymbol{m}}(\boldsymbol{u}_b) Y_{l\boldsymbol{m}}^*(\boldsymbol{u}_a) \tag{52}$$

where we eliminate ρ_n by choosing the gauge-fixed functional as an δ-functional, i.e. $\Phi[\rho] = \delta[\rho - 1]$. It is easy to check in the non-relativistic case, i.e. when $c \to \infty$, the energy spectra [1] of a particle near a sphere are recovered and given by

$$E = \pm \left(Mc^2 + \frac{\hbar^2[(D/2+l-1)^2 - 1/4]}{2MR^2} \right). \tag{53}$$

In (53), the rest energy Mc^2 and the minus sign are ignored in the non-relativistic theory. In $D = 3$ and 4, the formulae for (52) become the familiar representations

$$(\boldsymbol{u}_b|\boldsymbol{u}_a)_E = \frac{\hbar}{2Mc} \sum_{l=0}^{\infty} \frac{2Mc^2\hbar\mathrm{i}}{E^2 - \{M^2c^4 + (c\hbar/R)^2 l(l+1)\}} \frac{2l+1}{4\pi} \times \sum_{m=-l}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta_b) P_l^m(\cos\theta_a) \mathrm{e}^{\mathrm{i}m(\varphi_b - \varphi_a)} \tag{54}$$

and

$$(\boldsymbol{u}_b|\boldsymbol{u}_a)_E = \frac{\hbar}{2Mc} \sum_{l=0}^{\infty} \frac{2Mc^2\hbar\mathrm{i}}{E^2 - \{M^2c^4 + (c\hbar/R)^2[(l+1)^2 - 1/4]\}} \frac{l+1}{2\pi^2} \times \sum_{\substack{m_1=-l/2 \\ m_2=-l/2}}^{l/2} D_{m_1m_2}^{(l/2)}(\varphi_b, \theta_b, \gamma_b) D_{m_1m_2}^{(l/2)*}(\varphi_a, \theta_a, \gamma_a) \tag{55}$$

respectively, where $D_{m_1m_2}^{(l/2)}(\varphi, \theta, \gamma)$ are the representation functions of the rotation group. There are two corrections [1, 32] which must be added to bring the path integral from near to on the sphere. First the time-sliced action must measure the proper geodesic distance. For this, we have

$$A^N_{E,\mathrm{on}} = A^N_E + \Delta_4 A^N = \sum_{n=1}^{N+1} \left\{ \frac{MR^2}{\rho_n \epsilon_n}(1 - \cos\Delta\theta_n) + \frac{MR^2}{24\rho_n\epsilon_n}(\Delta\theta_n)^4 + \cdots \right.$$
$$\left. \cdots - \rho_n \epsilon_n \left(\frac{E^2}{2Mc^2} - \frac{Mc^2}{2} \right) \right\} \tag{56}$$

where $\Delta\theta_n$ is the small angle between $\boldsymbol{u}_n$ and $\boldsymbol{u}_{n-1}$. There is no need to go higher than quartic order since only the quartic term contributes to the relevant order ϵ in the limit $N \to \infty$. Let us calculate the $\Delta_4 A^N$ contribution in D dimensions. For very small ϵ, the fluctuations near the sphere will lie close to the $(D-1)$-dimensional tangent space. Let $\Delta\boldsymbol{x}_n$ be the coordinates in this space. Then we can write

$$\Delta_4 A^N \approx \frac{MR^2}{24\epsilon} \sum_{n=1}^{N+1} \left(\frac{\Delta \boldsymbol{x}_n}{R} \right)^4 . \tag{57}$$

The $\Delta\boldsymbol{x}_n$s have the lowest-order correlation $\langle \Delta x_i \Delta x_j \rangle_0 = (\hbar\epsilon/M)\delta_{ij}$. This shows that $\Delta_4 A^N$ has the expectation [1]

$$\langle \Delta_4 A^N \rangle_0 = \frac{\hbar^2}{2MR^2} \frac{(D^2-1)}{12} \sum_{n=1}^{N+1} \rho_n \epsilon_n \tag{58}$$

where $(D^2-1)/12$ is the contribution of the quartic term. The result is obtained by using the Wick contraction rules for the tensor $\langle \Delta x_i \Delta x_j \Delta x_k \Delta x_l \rangle_0 = (\epsilon\hbar/M) \times (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. This quantity will offer a correction in part. Second, the measure correction from the QEP for each short time in curved space without torsion is given by (26)

$$\Delta_{\mathrm{meas}} A^\epsilon_J = -\frac{\hbar}{6} \bar{R}_{\mu\nu} \Delta q^\mu \Delta q^\nu . \tag{59}$$

Here $\bar{R}_{\mu\nu}$ is the Ricci tensor which is $(D-2)g_{\mu\nu}/R^2$ for a sphere of radius R. The perturbative treatment of this correction gives the relevant contribution [1] to the energy

$$\langle \Delta_{\mathrm{meas}} A_J \rangle_0 = -\frac{\hbar^2}{6M} \frac{(D-1)(D-2)}{R^2} \sum_{n=1}^{N+1} \epsilon_n \rho_n . \tag{60}$$

When we add these two corrections to (47), the amplitude for a relativistic mass point on a sphere is

$$(\boldsymbol{u}_b|\boldsymbol{u}_a)_E = \frac{\hbar}{2Mc} \sum_{l=0}^{\infty} \frac{2Mc^2\hbar \mathrm{i}}{E^2 - \{M^2c^4 + (c\hbar/R)^2[l(l+D-2)]\}} \sum_{\boldsymbol{m}} Y_{l\boldsymbol{m}}(\boldsymbol{u}_b) Y^*_{l\boldsymbol{m}}(\boldsymbol{u}_a). \tag{61}$$

It is easy to check that the amplitude has the correct non-relativistic limit. We note that the amplitudes in (52) and (61) display the same wavefunctions for a particle moving on or near the sphere in spite of the particle being relativistic or not. This is because of the symmetry property of the sphere. Before closing this section it is worth stressing that by fixing to the sphere the relativistic invariance is violated.

4. The DK-equivalence between the project amplitude of a relativistic particle near a sphere with the Rosen–Morse potential

As in the non-relativistic case [1, 19], we can project the path integral of a relativistic particle near the surface of a sphere into a fixed azimuthal quantum number $\boldsymbol{m}$. There are similar angular barriers which arise in this projection. We can apply the DK-transformation to the relativistic project amplitudes of a particle near spheres in $D = 3$, 4 dimensions. The equivalence between the project amplitudes and the Rosen–Morse systems will be reproduced.

4.1. The DK-equivalence between the project amplitude of a relativistic particle near a sphere in $D = 3$ dimensions with the Rosen–Morse potential

In $D = 3$, the project amplitude of the fixed azimuthal quantum number m can be defined by [1]

$$(\boldsymbol{u}_b|\boldsymbol{u}_a)_E = \sum_m \frac{1}{\sqrt{\sin\theta_b \sin\theta_a}} (\theta_b|\theta_a)_{E,m} \frac{1}{2\pi} \, \mathrm{e}^{\mathrm{i}m(\varphi_b - \varphi_a)}. \tag{62}$$

Comparing (62) with (54), the projected amplitude can immediately be obtained by

$$(\theta_b|\theta_a)_{E,m} = \sqrt{\sin\theta_b \sin\theta_a} \sum_{l=m} \frac{\mathrm{i}c\hbar^2}{E^2 - \{M^2c^4 + (c\hbar/R)^2 l(l+1)\}} \frac{2l+1}{2} \frac{(l-m)!}{(l+m)!} \times P_l^m(\cos\theta_b) P_l^m(\cos\theta_a). \tag{63}$$

It is easy to find the λ-sliced path integral representation of this projective amplitude by comparing (62) and (47). It has the form

$$(\theta_b|\theta_a)_{E,m} \approx \frac{-\mathrm{i}\hbar}{2Mc} \int_0^\infty \mathrm{d}L \prod_{n=1}^{N+1} \left[\int \mathrm{d}\rho_n \, \Phi(\rho_n) \right] \frac{1}{\sqrt{2\pi\hbar\epsilon_b\rho_b/MR^2}} \times \prod_{n=1}^{N} \left[\int \frac{\mathrm{d}\theta_n}{\sqrt{2\pi\hbar\epsilon_n\rho_n/MR^2}} \right] \mathrm{e}^{-A_{E,m}^N/\hbar} \tag{64}$$

where

$$A_{E,m}^N = \sum_{n=1}^{N+1} \left\{ \frac{MR^2}{\epsilon_n\rho_n} [1 - \cos(\theta_n - \theta_{n-1})] - \hbar \log \tilde{I}_m(h_n) - \epsilon_n\rho_n \left(\frac{E^2}{2Mc^2} - \frac{Mc^2}{2} \right) \right\} \tag{65}$$

with h_n being defined by

$$h_n = \frac{MR^2}{\hbar\epsilon_n\rho_n} \sin\theta_n \sin\theta_{n-1}. \tag{66}$$

Later applications will require an analytic continuation of the path integral from integer values of m to arbitrary real values $\nu \geqslant 0$. For the arbitrary value ν, the relativistic amplitude can be calculated with the help of the addition theorem (e.g. [1])

$$I_\nu(\varsigma_n \sin\alpha \sin\beta) \, \mathrm{e}^{\varsigma_n \cos\alpha \cos\beta} = \sqrt{\frac{2\pi}{\varsigma_n}} \sum_{k=0}^\infty (k + \nu + 1/2) \frac{\Gamma(k + 2\nu + 1)}{k!} \times I_{k+\nu+1/2}(\varsigma_n) P_{k+\nu}^{-\nu}(\cos\alpha) P_{k+\nu}^{-\nu}(\cos\beta) \tag{67}$$

and the orthogonal relation

$$\int_{-1}^1 (\mathrm{d}\cos\theta) P_{k+\nu}^{-\nu}(\cos\theta) P_{k'+\nu}^{-\nu}(\cos\theta) = \frac{k!}{(k+2\nu)!} \frac{2}{2k + 2\nu + 1} \delta_{kk'} \tag{68}$$

where ν represents arbitrary real values. After some mathematical manipulations, the resulting amplitude is given by

$$(\theta_b|\theta_a)_{E,\nu} = \sqrt{\sin\theta_b \sin\theta_a} \sum_{n=0}^{\infty} \frac{\mathrm{i}c\hbar^2}{E^2 - \{M^2c^4 + (c\hbar/R)^2(n+\nu)(n+\nu+1)\}}(n+\nu+1/2) \times \frac{\Gamma(n+2\nu+1)}{n!} P_{n+\nu}^{-\nu}(\cos\theta_b) P_{n+\nu}^{-\nu}(\cos\theta_a). \tag{69}$$

This amplitude can be summed via a Sommerfeld–Watson transformation [33, 34] which leads to a simple closed-form expression (for $\theta_b > \theta_a$)

$$(\theta_b|\theta_a)_{E,\nu} = \frac{\hbar}{2Mc}\frac{-\mathrm{i}\mu}{\hbar}\Gamma(\nu - l(E))\Gamma(\nu + l(E)+1) P_{l(E)}^{-\nu}(-\cos\theta_b) P_{l(E)}^{-\nu}(\cos\theta_a) \tag{70}$$

where $\mu = MR^2$ and $l(E)$ is defined by

$$l(E) = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu}{\hbar^2}\left(\frac{E^2}{2Mc^2} - \frac{Mc^2}{2}\right)}. \tag{71}$$

With the analytic continuation ν of m, we observe that (64) has the continuum limit

$$(\theta_b|\theta_a)_{\mathrm{PT}}^{\mathrm{rel}} = \frac{-\mathrm{i}\hbar}{2Mc}\int_0^{\infty} \mathrm{d}L \int D\rho\, \Phi[\rho] \int D\theta\, \mathrm{e}^{-A_{\mathrm{PT}}^{\mathrm{rel}}/\hbar} \tag{72}$$

with the action

$$A_{\mathrm{PT}}^{\mathrm{rel}} = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \left[\frac{\mu}{2\rho(\lambda)}\theta'^2 - \rho(\lambda)\frac{\hbar^2}{8\mu} + \rho(\lambda)\frac{\hbar^2}{2\mu}\frac{\nu^2 - 1/4}{\sin^2\theta} - \rho(\lambda)\left(\frac{E^2}{2Mc^2} - \frac{Mc^2}{2}\right)\right] \tag{73}$$

where we have labelled the amplitude and action with rel and PT to denote that the problem is relativistic and the action has Pöschl–Teller potential [35, 36], respectively. As in the non-relativistic case [1], the action has a $1/\sin^2\theta$ singularity at $\theta = 0$ and $\theta = \pi$. Only with the property time-sliced action (65), this path integral is stable for all ν.

In the following we wish to prove the DK-equivalence between the Rosen–Morse system and the relativistic Pöschl–Teller system. First we take $f(\theta) = \sin^2\theta$ to proceed the f-transformation. This will remove the singularity in (73). The transformed action is given by

$$A_{\mathrm{PT}}^{\mathrm{rel,f}} = \int_0^S \mathrm{d}s \left\{\frac{\mu}{2\rho(s)\sin^2\theta}\dot{\theta}^2(s) - \frac{\rho(s)\hbar^2}{8\mu}\sin^2\theta + \frac{\rho(s)\hbar^2}{2\mu}(\nu^2 - 1/4) - \rho(\lambda)\left(\frac{E^2}{2Mc^2} - \frac{Mc^2}{2}\right)\sin^2\theta\right\}. \tag{74}$$

We now bring the kinetic term to the conventional form by solving the first-order differential equation $\mathrm{d}\theta/\mathrm{d}x = h'(x)$. The variable change is

$$\sqrt{f} = \sin\theta = h'(x) = \frac{1}{\cosh x} \tag{75}$$

which maps interval $\theta \in (0, \pi)$ into $x \in (-\infty, \infty)$. Therefore we have the effective potential

$$V_{\mathrm{eff}} = -\frac{\hbar^2}{M}\left[\frac{1}{4}\frac{h'''}{h'} - \frac{3}{8}\left(\frac{h''}{h'}\right)^2\right] = \frac{\hbar^2}{8\mu}\left(1 + \frac{1}{\cosh^2 x}\right). \tag{76}$$

The DK-transform of the action is given by

$$A_{\rm PT}^{\rm rel,DK} = \int_0^S {\rm d}s \left\{ \frac{\mu}{2\rho}\dot{x}^2(s) + \frac{\rho\hbar^2}{2\mu}\nu^2 - \frac{\rho}{\cosh^2 x}\left(\frac{E^2}{2Mc^2} - \frac{Mc^2}{2}\right)\right\}. \quad (77)$$

After choosing the gauge-fixed functional $\Phi[\rho]$ as a δ-functional, the action describes the motion of a non-relativistic particle in the smooth Rosen–Morse potential [36, 37]

$$V_{\rm RM}(x) = -\frac{\hbar^2}{2\mu}\frac{s(s+1)}{\cosh^2 x}. \quad (78)$$

Comparing the above equation with (77), we get the relation of the parameters between two systems

$$l(E) = s \quad (79)$$

$$\nu(E_{\rm RM}) = \sqrt{\frac{-2\mu E_{\rm RM}}{\hbar^2}}. \quad (80)$$

Therefore the DK equivalent relation is given by

$$(\theta_b|\theta_a)_{\rm PT}^{\rm rel,DK} = \frac{\hbar}{2Mc}\sqrt{\sin\theta_b\sin\theta_a}(x_b|x_a)_{E_{\rm RM}}. \quad (81)$$

From this we get the amplitude of the Rosen–Morse potential system

$$(x_b|x_a)_{E_{\rm RM}} = \frac{-{\rm i}\mu}{\hbar}\Gamma(\nu(E_{\rm RM}) - s)\Gamma(\nu(E_{\rm RM}) + s + 1)P_s^{-\nu(E_{\rm RM})}(\tanh x_b)P_s^{-\nu(E_{\rm RM})}(-\tanh x_a). \quad (82)$$

This result was given in [32] where the amplitude comes from the DK-transform of the non-relativistic system.

4.2. The DK-equivalence between the project amplitude of a relativistic particle near a sphere in $D = 4$ dimensions with the general Rosen–Morse potential

For a particle near a sphere in $D = 4$, the project amplitude of the fixed azimuthal quantum number m_1, m_2 can be defined by [1]

$$(\boldsymbol{u}_b|\boldsymbol{u}_a)_E = \sum_{\substack{m_1=-l/2\\ m_2=-l/2}}^{l/2} \frac{8}{\sqrt{\sin\theta_b\sin\theta_a}}(\theta_b|\theta_a)_{E,m_1,m_2}\frac{1}{2\pi}\,{\rm e}^{{\rm i}m_1(\varphi_b-\varphi_a)}\frac{1}{4\pi}\,{\rm e}^{{\rm i}m_2(\gamma_b-\gamma_a)}. \quad (83)$$

A comparison with (55) immediately gives the normalized projected amplitude

$$(\theta_b|\theta_a)_{E,m_1,m_2} = \sqrt{\sin\theta_b\sin\theta_a}\sum_l \frac{{\rm i}c\hbar^2}{E^2 - \{M^2c^4 + (c\hbar/R)^2[(l+1)^2 - 1/4]\}}\frac{(l+1)}{2}$$
$$\times d^{l/2}_{m_1,m_2}(\theta_b)d^{l/2}_{m_1,m_2}(\theta_a) \quad (84)$$

where l is summed from the larger value of $|2m_1|$, $|2m_2|$ to infinity. This amplitude and its path integral representation can be continued to arbitrary real values $m_1 = \mu_1$ and $m_2 = \mu_2$ with $\mu_1 \geqslant \mu_2 \geqslant 0$ [33]. Moreover, the path integral representation of the project amplitude can be obtained by using (47)–(50). It has the time-sliced form

$$(\theta_b|\theta_a)_{E,m_1,m_2} \approx \frac{-{\rm i}\hbar}{2Mc}\int_0^\infty {\rm d}L\prod_{n=1}^{N+1}\left[\int {\rm d}\rho_n\,\Phi(\rho_n)\right]\frac{1}{\sqrt{2\pi\hbar\epsilon_b\rho_b/(M/4)R^2}}$$
$$\times\prod_{n=1}^{N}\left[\int\frac{{\rm d}\theta_n}{\sqrt{2\pi\hbar\epsilon_n\rho_n/(M/4)R^2}}\right]{\rm e}^{-A^N_{E,m_1,m_2}/\hbar} \quad (85)$$

with

$$A^N_{E,m_1,m_2} = \sum_{n=1}^{N+1}\left\{\frac{MR^2}{\epsilon_n\rho_n}[1-\cos(\theta_n-\theta_{n-1})/2] - \hbar\log\tilde{I}_{|m_1+m_2|}(h^c_n) - \hbar\log\tilde{I}_{|m_1-m_2|}(h^s_n)\right.$$
$$\left. -\epsilon_n\rho_n\left(\frac{E^2}{2Mc^2}-\frac{Mc^2}{2}\right)\right\} \tag{86}$$

where the h^c_n and h^s_n are defined by

$$h^c_n = \frac{MR^2}{\hbar\epsilon_n\rho_n}\cos\theta_n/2\cos\theta_{n-1}/2 \tag{87}$$

$$h^s_n = \frac{MR^2}{\hbar\epsilon_n\rho_n}\sin\theta_n/2\sin\theta_{n-1}/2. \tag{88}$$

The analytic continuation of the amplitude (84) can be obtained by applying the folowing equality (e.g. [1])

$$I_{\mu_+}(\varsigma_n\cos\theta_n/2\cos\theta_{n-1}/2)I_{\mu_-}(\varsigma_n\sin\theta_n/2\sin\theta_{n-1}/2) = \frac{2}{\varsigma_n}\sum_{k=0}^{\infty}(2k+\mu_++\mu_-+1)$$
$$\times I_{2k+\mu_++\mu_-+1}(\varsigma_n)d^{k+\mu_1}_{\mu_1,\mu_2}(\theta_n)d^{k+\mu_1}_{\mu_1,\mu_2}(\theta_{n-1}) \tag{89}$$

with

$$\mu_1 \equiv \frac{\mu_++\mu_-}{2} \qquad \mu_2 \equiv \frac{\mu_+-\mu_-}{2} \tag{90}$$

and the orthogonal relation

$$\int_{-1}^{1}\mathrm{d}\cos\theta\, d^{k+\mu_1}_{\mu_1,\mu_2}(\theta)d^{k'+\mu_1}_{\mu_1,\mu_2}(\theta) = \frac{2}{2k+\mu_++\mu_-+1}\delta_{kk'}. \tag{91}$$

With a bit of mathematical treatment, the continuation solution of the relativistic amplitude (85) is found to be

$$(\theta_b|\theta_a)_{E,\mu_1,\mu_2} = \sqrt{\sin\theta_b\sin\theta_a}\sum_{n=0}^{\infty}\frac{\mathrm{i}c\hbar^2}{E^2-\left\{M^2c^4+(c\hbar/R)^2[(2n+2\mu_1+1)^2-1/4]\right\}}$$
$$\times\frac{(2n+2\mu_1+1)}{2}d^{n+\mu_1}_{\mu_1,\mu_2}(\theta_b)d^{n+\mu_1}_{\mu_1,\mu_2}(\theta_a). \tag{92}$$

As in the non-relativistic case [33], the sum over n can be performed by Sommerfeld–Watson transformation. It has the close form for $\theta_b > \theta_a$:

$$(\theta_b|\theta_a)_{E,\mu_1,\mu_2} = \frac{\hbar}{2Mc}\sqrt{\sin\theta_b\sin\theta_a}\frac{-\mathrm{i}\mu}{\hbar}\Gamma(\mu_1-l(E)/2)\Gamma(l(E)/2-\mu_1+1)$$
$$\times d^{l(E)/2}_{\mu_1,-\mu_2}(\theta_b-\pi)d^{l(E)/2}_{\mu_1,\mu_2}(\theta_a) \tag{93}$$

with $\mu = MR^2/4$ and

$$\frac{l(E)}{2} = -\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{4}+\frac{2\mu}{\hbar}\left(\frac{E^2}{2Mc^2}-\frac{Mc^2}{2}\right)}. \tag{94}$$

We note that (86) has the continuum limit

$$A^{\mathrm{rel}}_{\mathrm{PT'}} = \int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda\left\{\frac{MR^2}{8\rho(\lambda)}\theta'^2 - \frac{\rho(\lambda)\hbar^2}{8MR^2}+\frac{\rho(\lambda)\hbar^2}{2MR^2}\left[\frac{|\mu_1+\mu_2|^2-1/4}{\cos^2\theta/2}+\frac{|\mu_1-\mu_2|^2-1/4}{\sin^2\theta/2}\right]\right.$$
$$\left. -\rho(\lambda)\left(\frac{E^2}{2Mc^2}-\frac{Mc^2}{2}\right)\right\}. \tag{95}$$

Apart from the projected motion of a relativistic mass point near the surface of the sphere, this action, letting $\rho = 1$, also describes the dynamic of a particle in the general Pöschl–Teller potential [35, 36]

$$V_{\mathrm{PT}'}(\theta) = \frac{\hbar^2}{2\mu}\left[\frac{s_1(s_1+1)}{\sin^2\theta/2} + \frac{s_2(s_2+1)}{\cos^2\theta/2}\right]. \tag{96}$$

The subscript PT′ in the action of (95) denotes the above property. The superscript denotes that the particle is relativistic. We now introduce the auxiliary parameter $\mu = MR^2/4$. After rearranging the potential terms, we get the path integral representation of the project amplitude

$$(\theta_b|\theta_a)^{\mathrm{rel}}_{\mathrm{PT}'} = \frac{-\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}L \int D\rho\,\Phi[\rho]\int D\theta\,\mathrm{e}^{-A^{\mathrm{rel}}_{\mathrm{PT}'}/\hbar} \tag{97}$$

with the action

$$A^{\mathrm{rel}}_{\mathrm{PT}'} = \int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda\left[\frac{\mu}{2\rho(\lambda)}\theta'^2 - \rho(\lambda)\frac{\hbar^2}{32\mu} + \rho(\lambda)\frac{\hbar^2}{2\mu}\frac{\mu_1^2+\mu_2^2-1/4-2\mu_1\mu_2\cos\theta}{\sin^2\theta}\right.$$
$$\left. -\rho(\lambda)\left(\frac{E^2}{2Mc^2} - \frac{Mc^2}{2}\right)\right] \tag{98}$$

where the functional measure has the time-sliced form

$$\int D\theta \approx \frac{1}{\sqrt{2\pi\hbar\epsilon_b\rho_b/\mu}}\prod_{n=1}^{N}\left[\int\frac{\mathrm{d}\theta_n}{\sqrt{2\pi\hbar\epsilon_n\rho_n/\mu}}\right]. \tag{99}$$

Here we observe that the similar angular barriers in the non-relativistic case still survive. This is a natural result when we project the path integral near the surface of a sphere into the fixed azimuthal quantum numbers.

In the following, we wish to prove the equivalent relation between the relativistic Pöschl–Teller system of (97) and the general Rosen–Morse potential. Since $f(\theta) = \sin^2\theta$ one can remove the angular barrier. Therefore the f-transformed action is given by

$$A^{\mathrm{rel,f}}_{\mathrm{PT}'} = \int_0^S \mathrm{d}s\left[\frac{\mu}{2\rho(s)\sin^2\theta}\dot\theta^2(s) - \frac{\rho(s)\hbar^2}{32\mu}\sin^2\theta + \frac{\rho(s)\hbar^2}{2\mu}\right.$$
$$\left.\times[\mu_1^2+\mu_2^2-1/4-2\mu_1\mu_2\cos\theta] - \rho(s)\left(\frac{E^2}{2Mc^2}-\frac{Mc^2}{2}\right)\sin^2\theta\right]. \tag{100}$$

We now perform the h-transformation to transform the kinetic term into the conventional type by

$$h' = \sin\theta = \pm\frac{1}{\cosh x} \qquad \cos\theta = -\tanh x. \tag{101}$$

We note that the interval $\theta \in (0, \pi)$ is mapped into $x \in (-\infty, \infty)$ again. From (76) the effective potential is

$$V_{\mathrm{eff}} = \frac{\hbar^2}{8\mu}\left(1 + \frac{1}{\cosh^2 x}\right). \tag{102}$$

Therefore from (100) we get the DK-transformation

$$A^{\mathrm{rel,DK}}_{\mathrm{PT}'} = \int_0^S \mathrm{d}s\left\{\frac{\mu}{2\rho}\dot x^2(s) + \frac{\rho\hbar^2}{2\mu}[\mu_1^2+\mu_2^2+2\mu_1\mu_2\tanh x] - \frac{\rho}{\cosh^2 x}\right.$$
$$\left.\times\left[\left(\frac{E^2}{2Mc^2}-\frac{Mc^2}{2}\right) - \frac{3\hbar^2}{32\mu}\right]\right\}. \tag{103}$$

This action, choosing $\rho = 1$, has the general Rosen–Morse potential

$$V_{\mathrm{RM}'}(x) = \frac{\hbar^2}{2\mu}\left[-\frac{s(s+1)}{\cosh^2 x} + 2A\tanh x\right] \tag{104}$$

with the relation of the parameters between two systems

$$\left(\frac{E^2}{2Mc^2} - \frac{Mc^2}{2}\right) - \frac{3\hbar^2}{32\mu} = \frac{\hbar^2}{2\mu}s(s+1) \tag{105}$$

$$\mu_1\mu_2 = A \tag{106}$$

$$E_{\mathrm{RM}'} = -\frac{\hbar^2}{2\mu}\left[\mu_1^2 + \left(\frac{A}{\mu_1}\right)^2\right]. \tag{107}$$

Applying (105) to (94), we determine the third parameter relation

$$l(E)/2 = s. \tag{108}$$

Finally we obtain the amplitude of the general Rosen–Morse system by

$$(\theta_b|\theta_a)^{\mathrm{rel.DK}}_{\mathrm{PT}'} = \frac{\hbar}{2Mc}\sqrt{\sin\theta_b\sin\theta_a}(x_b|x_a)_{E_{\mathrm{RM}'}}. \tag{109}$$

Explicitly, the amplitude can be expressed by

$$(x_b|x_a)_{E_{\mathrm{RM}'}} = \frac{-\mathrm{i}\mu}{\hbar}\Gamma(\mu_1 - s)\Gamma(s - \mu_1 + 1)d^s_{\mu_1,-\mu_2}(\theta_b(x) - \pi)d^s_{\mu_1,\mu_2}(\theta_a(x)) \tag{110}$$

with the variables relation

$$\cos\theta = -\tanh x \qquad \theta \in (0, \pi) \qquad x \in (-\infty, \infty). \tag{111}$$

This result was also obtained in [33] from the non-relativistic DK-transformation.

5. Summary

In this paper we have calculated the path integrals of a relativistic particle near and on a sphere. As an application, we get the amplitudes of Rosen–Morse and general Rosen–Morse systems by applying the DK-transformation to the relativistic project amplitudes of a particle near the surface of spheres in $D = 3$, 4 dimensions. From this application, we observe that the path integrals of some relativistic potential problems, controlled by Klein–Gordon equation in operator level, can also be solved by DK-transformation. There are two interesting problems which relate to a relativistic particle on a sphere in four-dimensional Euclidean space. First, a relativistic particle moves on an $SU(2)$ group. We immediately obtain, owing to the $SU(2) \approx S^3$, the relativistic path integral on the group $SU(2)$. Second, the difference between $SU(2)$ and the spinning top is that the $SU(2)$ is two points covering the space of the spinning top [1]. On the spinning top, the Euler angles γ and $\gamma + 2n\pi$ are physically indistinguishable, where n is an arbitrary integer. The amplitude must reflect this property. There is a simplest possibility. It is given by

$$(\varphi_b, \theta_b, \gamma_b|\varphi_a, \theta_a, \gamma_a)^{\mathrm{rel}}_{E,\mathrm{top}} = (\varphi_b, \theta_b, \gamma_b|\varphi_a, \theta_a, \gamma_a)^{\mathrm{rel}}_{E,SU(2)} + (\varphi_b, \theta_b, \gamma_b + 2\pi|\varphi_a, \theta_a, \gamma_a)^{\mathrm{rel}}_{E,SU(2)}. \tag{112}$$

Here the normalized amplitude $(\varphi_b, \theta_b, \gamma_b|\varphi_a, \theta_a, \gamma_a)^{\mathrm{rel}}_{E,SU(2)}$ is defined by [1]

$$(\boldsymbol{u}_b|\boldsymbol{u}_a)^{\mathrm{rel}}_{E,\mathrm{sphere}} = \frac{1}{2\pi}(\varphi_b, \theta_b, \gamma_b|\varphi_a, \theta_a, \gamma_a)^{\mathrm{rel}}_{E,SU(2)}. \tag{113}$$

The sum in (112) eliminates all half-integer representation functions $D^{l/2}_{mm'}$ in the expansion (55) of the amplitude.

From the viewpoint of the exactly solvable problems [31, 38], the relativistic problems discussed in this paper can be integrated into the group path integral even though they are controlled by the Klein–Gordon equation in the operator level. A more unified description for the classification of the path integral solutions is achieved.

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